Learning Math as a Second Language

A Journey to Mastery

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Abstract

Mathematics is, by nature, a very technical subject. Assessments focus on students' ability to solve problems that require a variety of techniques. But what of comprehension and fluency in mathematics? Our ability to develop mathematical understanding at these deeper levels depends on our willingness to let go of our reliance on procedural instruction and instead teach towards mathematical literacy. Using the topic of square-roots, we delve into what this might look like in the classroom.

Introduction

Imagine a young child, say three or four years of age. This child has a favorite book, which she requests to be read to her before bed every night. Over time, the child learns the story well enough to read along with the book. To the casual observer, one might credit this child with more skill at reading than is actually the case. Provided with a different book, one would see that the child may be able to identify a few words here and there, but her actual reading skills may be significantly lower than initially assumed.

In mathematics, high-stakes standardized exams are the equivalent of this child's bedtime story. It does not take long for instructors to recognize the problems that appear more frequently, and thereby arrange their curriculum to focus on those specific problems that would improve their students' outcomes on these assessments.

It is all too common that students are taught to identify specific types of problems and implement a predetermined step-by-step process whenever they've recognized a particular type of problem. In essence, students are taught the story, which they learn to recite from memory. This approach can be very effective at helping students earn good scores from an exam whose content is known in advance. This approach is also quite efficient at covering a wide range of material, as students are not necessarily required to absorb the meaning of what they're doing - so long as they can perform the specific tasks that they're given.

Throughout this process, students learn implicitly that the subject of mathematics is about knowing the necessary procedures to solve various problems. They may even learn that what they write doesn't have to make sense to them - because someone else is always there to tell them whether or not they are correct. If they can just learn enough processes, they'll pass the class and hopefully they'll never have to do math again in their life. These students end up robbed of the deeper ideas that form the foundation of why mathematics is a core subject in the first place.

There are, however, a handful students for whom the abstractions of mathematics come more naturally - and these students may be able to piece together a bigger picture of the subject. They may even pursue mathematics as a career; thereby continuing the cycle that mathematics is a subject that is learned by mimicking processes and procedures until a deeper notion forms. In other words, math is a subject for "math people" - individuals who are capable of independently abstracting the language of mathematics from myriad examples.

Imagine if elementary school students were subjected to reading classes whose curriculum consisted only of having book after book read *to* them until they developed their own ability to read. While this might work for a handful of students, we would never accept that there are "reading people" and then there are other students for whom reading just never made sense.

1 Identifying the Bottleneck

Math instructors frequently encounter student work that has a structure resembling correct work, with incorrect content. To an expert, it may seem baffling that the structure for a correct result is present, and somehow, despite the structure, the student arrives at an incorrect answer. I like to think of this as "Math Libs"[1] syndrome. This issue frequently occurs when students have been taught an algorithm for solving a given type of problem. The algorithm takes the form of a Mad Lib for the student, into which they place the particulars of their specific problem. The root of the issue is that students often do not understand the overall meaning of their work, resulting in an incorrect "filling in of the blanks" for the algorithm.

This bottleneck, stemming from a lack of "understanding"[2], has been known and written about for decades. In 1976, Richard Skemp outlined the difference between "instrumental understanding" and "relational understanding" citing examples from textbooks as a major source of students' reliance on the former[17]. What I've termed as "Math Libs", he called "rules without reason", but the meaning is the same - students may know *what* to do, but not *why*. By relying on instrumental understanding, students are left unable to confirm their results, or process whether or not they have an answer that "makes sense."

For the purposes of this article, we will focus on students' abilities for the specific topic of working with square-roots. We begin by analyzing a piece of incorrect student work.

Sample student work:

$$\sqrt{75} \to \sqrt{3 \cdot 25} \to 3\sqrt{5}$$

In this example, the student has performed the "traditional" step of factoring the radicand[3] to display a perfect-square factor. However, in the second step, $\sqrt{25}$ is identified to be 5, but the positions of 3 and 5 have been inexplicably reversed. To an expert, this second step would obviously be false - so what is the student thinking?

Let's identify what the student might know, based on this work. First, in factoring 75 as the product of 25 and 3, the student has identified that perfect-squares are relevant to the simplification of square-root expressions. Second, the student has identified 5 as the square-root of 25 (presumably so, since 25 does not appear in their final answer - despite the fact that 5 ended up as a radicand).

What the student does not understand here is how the various pieces of this process fit together. In other words, they are lacking a bigger perspective of this

[1] `https://en.wikipedia.org/wiki/Mad_Libs`

[2] Skemp classified the word "understanding" as a *faux ami*, akin to homonymous, potentially having different meanings to different people

[3] "Radicand" is the formal word to describe *what* we are taking the root *of*. This applies to all radicals, not just square-roots.

topic, and as a result, they are lost in trying to string individual steps together into a coherent narrative.

Furthermore, it is clear that this student is lacking in their ability to validate their own results. If we are aiming to develop mastery in our students, we must also build their self-sufficiency, freeing them from the notion that the teacher is the only one qualified to verify their answers.

It would be inappropriate to teach this student to simply interchange the 3 and the 5 in their answer. The student has displayed a lack of relational understanding regarding square-roots (and what they represent), and what's missing is not more procedures. The structure of the student's work shows that they're already following (and mostly succeeding at) a well-established procedure. If this student is to learn how to simplify square-roots consistently and accurately, they must develop a deeper idea of what these symbols *represent*.

Consider how we might treat a student who wrote the sentence: "The lazy dog jumped over the quick brown fox." The writing of this sentence ensures that the student knows how to write all of the letters in the alphabet. The student also knows something about the structure of sentences, capitalizing their first letter, and ending with a period. However, there seems to be a disconnect in the meaning of the sentence itself. It is as though the student is attempting to replicate something they've seen before, but there is something lost in the order of the words. We could chalk this error up to poor recollection, but more fundamentally, it would seem that the student did not understand what was originally being conveyed in the classic sentence: "The quick brown fox jumped over the lazy dog." The fox is quick, giving it the speed and agility to jump over the lazy dog, who is much more likely to be stationary. This student has correctly recalled all the nouns, adjectives, and verbs - but then even the order of these objects becomes an exercise in recollection.

In treating mathematics as a language, we recognize that there is meaning behind the symbols we appear to be manipulating. We must convey to our students that math is more than just the surface-level manipulations that appear on our chalkboards - it is the expression of relationships between values[4]. If our students do not learn to read the symbols and distill the meaning behind them, they cannot truly apply their knowledge of mathematics outside the classroom. And for those students with excellent number sense, learning to express their internal processes in formal mathematical language will aid them in advancing their skills to handle more complex problems.

[4]Even between values that are known and unknown, in the case of algebra.

2 How Does an Expert Handle It?

No student begins their journey as an expert, just as the instructor similarly did not start out as an expert either. So it makes sense to first ask, ”how did the expert arrive at their level of mastery?” Hoffman described the path to mastery as a continuum, progressing from novice to master [14]. His view of mastery is based on a *relative* approach to expertise, where all participants are seen as progressing towards mastery, rather than a *fixed* approach, where mastery is reserved for those with the necessary predisposition [7]. The fixed view of mastery is very common in mathematics - as evinced by the ubiquitous belief in the existence of ”math people”. In order to develop expert thinking in our students, it is imperative that we both adopt and promote the relative approach, believing in all of our students' ability to progress towards mastery.

Applied to the context of students and their mathematical problem-solving skills, the following interpretation of Hoffman's original continuum of mastery is proposed:

Acclimation	**Novice**	Is able to interpret symbols and terminology according to their primary definition.
	Initiate	Solves problems ”the long way”, by breaking them down to their most fundamental level and applying the definition in order to find solutions.
Competence	**Apprentice**	Recognizes patterns while applying the definition during problem-solving and begins ”skipping steps”, forming their own process. This pattern recognition begins to extend beyond the strict boundaries of the problem at hand, forming connections with other tangentially connected topics.
	Journeyman	Has developed their own streamlined process to consistently solve ”recognizable”[5] problems and justifies their results verbally or in full-sentences. Understands the implications of their process in the broader context of related topics.
Proficiency	**Expert**	Is capable of adapting their own process to tackle ”new”[6] problems effectively. Thoroughly answers questions about modifications to their process with appropriate vocabulary and solid reasoning.
	Master	Comprehends a variety of processes, selecting an optimal approach dependent on the problem being solved.

[5] We use the word ”recognizable” to describe those problems for which the student has

With these stages of mastery development in mind, let's return to the sample student work provided in the previous section. What does a master see in this problem that the sample student does not?

Even to a Novice, the initial expression, "$\sqrt{75}$", should have a very specific meaning:

$$\sqrt{75} \text{ is the positive value whose square is 75.}$$

We say that even the novice should have this understanding because this is a literal interpretation of "the square-root of 75". Both the Initiate and the Apprentice should be capable of recognizing that 75 is not a *perfect*-square; that is to say, 75 is not the square of any whole number. As such, the Apprentice might further understand that $\sqrt{75}$, as a non-perfect-square, must represent an irrational[7] value. As an irrational number, $\sqrt{75}$ has an infinitely long decimal representation, forcing "$\sqrt{75}$" to be the only *exact* way to refer to this specific number. The Journeyman also understands that any decimal representation of this value will be merely an approximation (whose square is not expected to be exactly 75).

On a broader level, the Journeyman understands that there can be many different ways to express the same value. For instance, $\frac{1}{2}$, $\frac{3}{6}$, and 0.5 all express the same value despite their cosmetic differences. In the specific case of $\sqrt{75}$, the Journeyman knows that any representation, when squared, must be 75. After all, this is equivalent to the definition of $\sqrt{75}$ that was given at the beginning of this section. With this level of understanding, it is clear that $3\sqrt{5}$ cannot be equivalent to $\sqrt{75}$ because the square of $3\sqrt{5}$ is 45 rather than 75:

$$(3\sqrt{5})^2 \rightarrow (3\sqrt{5})(3\sqrt{5}) \rightarrow 3 \cdot 3 \cdot \sqrt{5} \cdot \sqrt{5} \rightarrow 3 \cdot 3 \cdot 5 \rightarrow 45$$

The sample student work provided is very similar to what an instructor might write as their "steps" for simplifying $\sqrt{75}$. The fundamental difference is that the Journeyman understands the confirmation of their result through a process similar to how we saw that the square of $3\sqrt{5}$ was 45 instead of 75:

$$(5\sqrt{3})^2 \rightarrow (5\sqrt{3})(5\sqrt{3}) \rightarrow 5 \cdot 5 \cdot \sqrt{3} \cdot \sqrt{3} \rightarrow 5 \cdot 5 \cdot 3 \rightarrow 75$$

Now, because they have verified that $5\sqrt{3}$ has 75 as its square, they can see explicitly that $5\sqrt{3}$ is "a number whose square is 75", and therefore must be equivalent in value to $\sqrt{75}$ (despite the cosmetic differences). If our example student had this level of comprehension, they could have determined on their own that $3\sqrt{5}$ had 45 as its square, meaning that their result was equivalent to $\sqrt{45}$, rather than the $\sqrt{75}$ they were originally looking to simplify. This level of understanding enables the student to verify their own work, rather than needing to wait for external validation.

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already had extensive practice.

[6] We use the word "new" to describe those problems for which the student has little to no prior exposure.

[7] "Irrational" refers to values with decimal representations that neither terminate (such as 0.25) nor have repeating pattern in their digits (such as $0.33\overline{3}$).

It should be noted that the problem used in our example represents a very common problem-type for students in Pre-algebra and Algebra. As such, it is difficult to address the further capabilities of an Expert or Master in the context of this problem. Problems requiring innovative solution methods, such as real-world applications, would be a better gauge of how an Expert or Master would apply their understanding.

3 Modeling For Students

In preparing to model new concepts for our students, it is important to consider the progression of conceptual development that we are intending for our students. If our goal is for students to develop a deep *relational* understanding of the topic, we must provide them with the necessary opportunities to both explain their current ideas as well as question the possible directions for extending those ideas.

Hiebert and Carpenter contended that the development of this type of relational understanding is rooted in forming connections between facts, procedures and ideas; building primarily on students' prior knowledge and experiences[12]. Understanding is constructed rather than received; it promotes remembering, while simultaneously reducing the amount of information that needs to be remembered; and it enhances the ability to transfer knowledge to new situations in order to solve new problems. However, "when students do not understand, they perceive each topic as an isolated skill, and they cannot apply their skills to solve problems not explicitly covered by instruction, nor extend their learning to new topics." [5] "The mark of powerful learning is the ability to solve problems in new contexts or to solve problems that differ from the ones one has been trained to solve."[16]

Within this context it is important to be aware that teaching the conventional algorithms too early risks students missing out on relational knowledge while gaining instrumental knowledge, resulting in a loss of faith in their own understanding of numbers and an inflexibility in problem solving, relying instead on a single procedure without thought [8]. Pushing students' instrumental understanding beyond their relational understanding can lead to rote memorization and prevent them from gaining the sort of deep relational understanding of the topic that is necessary for mastery [6].

In order to develop *expert thinking* within our students, we must begin by being willing to do things "the long way". Experts in mathematics often prioritize speed and efficiency when showing work on a problem. Reaching the correct answer in as few steps as possible is too often the goal - leaving all but the best students behind, struggling to figure out what happened between the lines. On the other hand, erring on the side of showing "too much" work may initially frustrate some of the speedier students, but it still serves to emphasize the thought processes that are often not explicitly communicated on paper.

Let's first look at how square-roots are initially defined. In both textbook and classroom lectures, it is frequently the case that square-roots are initially defined by how square-roots and perfect-squares interact. However, it must be stressed from the beginning *what the symbols mean.* We cannot neglect the importance of starting out at the Novice level, ensuring that all students gain the necessary foothold for comprehension of square-roots. What follows is just an *outline* for instructors seeking to lead students to a deeper understanding of square-roots. Students will initially struggle to communicate their thoughts about mathematics, and as such it is recommended that this approach to mathematical literacy be used comprehensively throughout an entire term of instruction.

> Instructor: (*eventual student response.*)
>
> "The square-root of nine" is another way of saying "the number whose square is nine". Who can give me a number whose square is nine? *(three.)* And why is it that "three" is the square-root of "nine"? *(Because the square of three is nine.)*
>
> What about the square-root of 25? What does "the square root of 25" refer to? *(the number whose square is 25.)* And who can give me a number whose square is 25? *(five.)* Remind me why we can say that "five" is the square-root of 25? *(Because the square of five is 25.)*

In the sample classroom interaction described above, no written symbols are used - the discussion is purely descriptive in nature. Only after students have been introduced to the *idea* of square-roots as a concept should the mathematical symbols be introduced. This is because the mathematical symbols are merely shorthand for the concepts that were described in the discussion. Students need to have the descriptive background in place before introducing the symbolic notation; because otherwise, what is it that these symbols are symbolizing? In putting the conceptual definition of square-roots at the forefront of our instruction, we ensure that all students are starting with an introduction to the Novice-level of understanding.

When it comes to introducing the symbolic notation, it should be stressed to students that the symbolic square-root and its simplification are equivalent because they represent the same value.

> $\sqrt{36}$ is the symbolic representation for the square-root of 36. Who can tell me what "the square-root of 36" refers to? *(the number whose square is 36.)* Good. So we must be saying that:
>
> $$\sqrt{36} \cdot \sqrt{36} = 36$$
>
> Who can give me a number whose square is 36? *(six.)* Very good. Why can we say that "six" is the same as "the square-root of 36"? *(Because the square of six is 36.)* I see. So what I hear you saying is that:
>
> $$6 \cdot 6 = 36$$

[to be written in alignment with the previous symbolic notation.]

So, what we're seeing here is that BOTH "$\sqrt{36}$" AND "6" have a square of 36. What do you suppose this means about "$\sqrt{36}$" and "6"? *(they are equal / they represent the same value.)*

At this point the instructor has explicitly demonstrated several of the ways an "expert" understands square-roots: by description according to the definition; by confirming that the value of the square matches the initial radicand; and by identifying different-looking values as equivalent. Most importantly, all of this has taken place within the comfortable context of perfect-square radicands. By taking this approach, we are pushing students to develop the formal language to describe behavior that they are hopefully already beginning to understand. We are also laying the groundwork for students to grow into the Apprentice- and Journeyman-levels of understanding. And now that we've laid the groundwork, it's time to push the concept of square-roots into the realm of irrational numbers.

$\sqrt{15}$ is the symbolic representation for the square-root of 15. Who can tell me what "the square-root of 15" refers to?[8] *(the number whose square is 15.)* So, what does that tell us about "the square-root of 15"? *(its square is 15.)* What do you think we'll get if we multiply: $\sqrt{15} \cdot \sqrt{15}$?[9] *(fifteen.)* Can anyone give me a number whose square is 15?

We pause here to note that students are not expected to provide a number whose square is 15. Fifteen is not a perfect-square, and as an expert, we know that "the number whose square is 15" has an infinite decimal representation. It would be impossible to write it down in its entirety. However, as we've only been dealing with perfect-squares, whose square-roots are whole numbers, students are apt to respond, "there is no number whose square is 15." If calculators are available, the instructor should invite students to see what their calculators have to say about that. Students will then provide the instructor with a decimal approximation for $\sqrt{15}$, at which point the instructor should ask:

What's so special about this decimal number? *(its square is 15.)* That's right, this is the number whose square is 15 - let's use our calculators to square this number and make sure that we get 15. *(students recognize that the square is not exactly 15.)* Why didn't we get 15 when we squared this number? *(we didn't use all the digits from our calculator.)*[10]

The instructor now has a perfect opportunity to discuss irrational numbers and the closely-related topic of decimal approximation. This discussion serves

[8]Instructor gives no hint or suggestion that this is in any way different from previous perfect-square examples. Instructor asks the same question about square-root of 16 if students are struggling to respond, then returns to the question about square-root of 15.

[9]We're also reinforcing what "squaring" looks like, when written out.

[10]Students may need some guiding questions to come to this conclusion.

to further ground students understanding of square-roots in the concrete realm of decimal numbers.

> The decimal value for $\sqrt{15}$ goes on forever, beyond what the calculator can display - even the calculator must round the value off at some point! And because we rounded, our square cannot be 15, because we're now squaring a different value - even though the difference is very small! So, whenever we round a decimal number, we will use the $\approx$ symbol to indicate that we're not *exactly* equal because we rounded.
>
> $$\sqrt{15} \approx 3.87298334621$$
>
> Does this decimal number make sense as the value for $\sqrt{15}$? Where can we find 3.87298334621 on the number line? *(between 3 and 4.)* And does that make sense for $\sqrt{15}$? *(yes, because $\sqrt{15}$ is between $\sqrt{9}$ and $\sqrt{16}$.)*[11] Which are we closer to: $\sqrt{9}$ or $\sqrt{16}$? So should our decimal be closer to 3 or 4?

If symbolic calculators are available, have students compute the square of $\sqrt{15}$ symbolically, instead of simply squaring the decimal approximation. This should reinforce the idea that the symbolic representation is the *exact* representation. This allows the instructor to motivate the use of symbolic notation, in that it is the only truly accurate way to express this "bizarre" value.

At this point students should be able to describe, using full sentences, what the square-roots of various radicands represent. Students should also be able to justify why, for example, four is the square-root of 16 - again, using full sentences. And when asked for an explicit decimal representation of a square-root expression, students should be capable of describing why the decimal approximation for a square root doesn't give "the right" value when squared.

Our next area of focus should be on confirming the equivalence of different representations of the same square-root. This is an often-overlooked skill in the context of teaching square-roots. However, it represents a fundamental technique in an expert's toolkit - supporting the recognition of equivalent representations of the same value. By way of example, we will guide students through the process of recognizing the equivalence of $\sqrt{12}$ and $2\sqrt{3}$:

> Who can tell me what $\sqrt{3}$ represents? *(the number whose square is three.)* Who can tell me what $2\sqrt{3}$ represents? *(double the number whose square is three, or $\sqrt{3}+\sqrt{3}$.)*
>
> What do we know about $\sqrt{3}$? *(its square is three.)* (Symbolically reaffirm by writing: $\sqrt{3}\cdot\sqrt{3}=3$.) Do we know anything like that about $2\sqrt{3}$?

The instructor should then guide students through the process of symbolically computing the square of $2\sqrt{3}$:

$$(2\sqrt{3})^2 \to (2\sqrt{3})(2\sqrt{3}) \to 2\cdot 2\cdot\sqrt{3}\cdot\sqrt{3} \to 2\cdot 2\cdot 3 \to 12$$

[11] Students may need some guiding questions to reach this conclusion.

This may be preceded (or anteceded) by a computation with the squaring of a decimal approximation for $2\sqrt{3}$. In both cases, students should be guided towards the conclusion that $2\sqrt{3}$ is equivalent to $\sqrt{12}$ since both values represent "the number whose square is 12". Furthermore, direct approximation of both $2\sqrt{3}$ and $\sqrt{12}$ support the conclusion that both representations have the same value.

The instructor may want to simultaneously discuss the equivalence of expressions such as $\sqrt{3}\cdot\sqrt{7}$ and $\sqrt{21}$ in a similar fashion. (Both expressions represent "the value whose square is 21.) This is a nearly identical concept to that of the "mixed radical" variety, but it should not be taken for granted that all students will immediately recognize the similar conclusion. This specific notion of "multiplication or factoring of radicals" forms an essential piece of the puzzle for the simplifying of square-root expressions.

At this point, students should be able to identify an equivalent square-root for any given mixed radical. Students should also be able to write a full sentence describing *why* the two values are equivalent[12].

With all of the background material now in place, students should be ready to more fully comprehend the process of simplifying square-root expressions. If this process is taught without the prior discussions, as is often the case, many students will be left lacking the necessary understanding to conceive of their work in any relevant context. In other words, they may learn how to perform the instrumental task of simplifying a square-root expression, but they will not have the relational understanding needed in order to apply the idea of square-roots to any real-world task.

> We've now seen the equivalence, and hence the interchangeability of expressions such as $\sqrt{3}\cdot\sqrt{7} \leftrightarrow \sqrt{21}$ and $2\sqrt{3} \leftrightarrow \sqrt{12}$. Our goal now is to distinguish which square-root expressions are "simplify-able" and which are not.
>
> For example, $\sqrt{12}$ is simplify-able because it is equivalent to an integer multiple of a square-root with smaller radicand. However, radicals such as $\sqrt{21}$ are *not* simplify-able because its factorization, $\sqrt{3}\cdot\sqrt{7}$ is a product of *radicals*, neither of which is a "nice" integer.
>
> If you'll recall, we found $\sqrt{12}$ *after* we started with $2\sqrt{3}$ and then found its square. But can we go the other way? Can we start with the square-root of a large number, and simplify it down to a multiple of a smaller square-root? Think of our equivalence between $\sqrt{21}$ and $\sqrt{3}\cdot\sqrt{7}$, can we do something similar to $\sqrt{12}$? *(Students either suggest breaking up into $\sqrt{6}\cdot\sqrt{2}$ or $\sqrt{3}\cdot\sqrt{4}$.)* Prompt students to continue by breaking down the remaining square-root of a compound number to end up at $\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{3}$. Can we break down any of these square-roots further? *(no.)* Why not? *(because all radicands are prime numbers.)*[13]

[12] They are equivalent because both values represent "the number whose square is #".

[13] this explanation will be much easier if the instructor has made regular use of the Fundamental Theorem of Arithmetic.

> Can anyone see a way to "simplify" $\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{3}$? *(the square of "the square-root of two" is two.)* Who can explain why this simplification is possible? *(The square-root of two is "the number whose square is two". Therefore, when we square "the square-root of two", the result must be two.)* So, if $\sqrt{2} \cdot \sqrt{2}$ is 2, then $\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{3}$ is $2 \cdot \sqrt{3}$.
>
> Let's try a new one together. See if we can simplify $\sqrt{18}$. Remember, keep factoring your radicand until all of your radicands are prime numbers. [Students work independently/with a partner.] Who completely factored $\sqrt{18}$? What did you get? *($\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{3}$)* Very good. Who sees how this can be simplified? *($\sqrt{2} \cdot 3$ or $3\sqrt{2}$)*

By requiring students to completely break down their square-root expression into a product of "prime" radicals and emphasizing the meaning of each individual radical as "the number whose square is *the radicand*", students not only have a clear process, but they should be confident in their grasp of the idea behind it as well. Furthermore, this approach scales very easily to include the simplification of algebraic square-root expressions as well. For example:

$$\sqrt{98x^3} \rightarrow \sqrt{2}\sqrt{7}\sqrt{7}\sqrt{x}\sqrt{x}\sqrt{x}$$

$$\sqrt{2}\sqrt{7}\sqrt{7}\sqrt{x}\sqrt{x}\sqrt{x} \rightarrow \sqrt{2}(\sqrt{7}\sqrt{7})(\sqrt{x}\sqrt{x})\sqrt{x}$$

$$\sqrt{2}(\sqrt{7}\sqrt{7})(\sqrt{x}\sqrt{x})\sqrt{x} \rightarrow \sqrt{2} \cdot 7 \cdot x \cdot \sqrt{x} \rightarrow 7x\sqrt{2x}$$

It would also be prudent, at this point, to explain to students why $3 \cdot \sqrt{2}$ is preferable to $\sqrt{2} \cdot 3$. Because multiplication often goes *un-written*, it can be confusing as to what the radicand might be in the case of $\sqrt{2} \cdot 3$. Someone who is not careful enough might mistake $\sqrt{2} \cdot 3$ for $\sqrt{2 \cdot 3}$. Imagine how much easier it is to make this mistake when we aren't writing $\cdot$ for multiplication.

It is also important for the instructor to elicit from students an idea of what makes a square-root expression "simple" or "simplified". Students should be guided to the conclusion that a square-root expression is "simplified" when we have expressed it with the smallest possible radicand. Asking students *why* we would bother with "simplifying" these square-root expressions can also be fruitful. The answer[14] to this question further motivates an understanding of the process as well as the more general context of square-roots.

At this point, it is up to each individual student to begin refining their process for this task of simplification. We've led all students up through the Initiate stage: solving problems "the long way" by breaking the problem down to its most fundamental level. Students, on their own, must begin to recognize common patterns and they may begin to "skip" steps in the process we've outlined. This should be encouraged, and these students should be asked to

[14] While we have seen that there can be many representations of the same square-root value, there is only *one* "simplified" representation. This forms an easy basis for comparison of values that have a variety of representations. The same holds true in other areas of mathematics, as in the case of reducing fractions.

explain their "skipped" steps when publicly sharing solutions with their peers. Through this exercise of justifying themselves, the student begins progressing through the Journeyman stage - as their observing peers are introduced to patterns of thinking that they themselves may not have yet realized. Furthermore, we are recognizing and supporting diversity in individual understanding, which is a fundamental part of learning [10]. In so doing, we are appropriately placing correctness, precision, prompt recall and speedy task completion as secondary to the individual's process of making sense of the mathematics [2, 3].

4 Practice and Feedback

In keeping with the notion that mathematics should be taught as a language, students should be consistently asked to write full sentences describing the symbolic manipulations that they are being asked to perform.

For example:

- "Explain why '7' and '$\sqrt{49}$' are equivalent."
- "Your friend is unsure of where '$\sqrt{17}$' belongs on the number line. How would you explain it to them?"
- "Find an equivalent expression for '$6\sqrt{2}$' and explain why your answer represents the same value."
- "Is '$\sqrt{75}$' the same as '$3\sqrt{5}$'? Why or why not?"
- "Your friend is trying to simplify '$\sqrt{40}$' and they got stuck at '$\sqrt{8} \cdot \sqrt{5}$'. How would you explain what they should do next?"
- "How many different ways can you find to represent 'the square-root of 72'? Explain why each of your expressions is a representation of the square-root of 72."

Of course, student practice should also include the usual types of computational problems for this topic. After all, students will still be responsible for completing these sorts of problems, exclusively of the computational variety, on standardized assessments. However, including questions that are descriptive in nature not only allows the instructor to assess procedural proficiency, but conceptual fluency as well.

Feedback should prioritize the addressing of misconceptions over procedural mistakes. Students' procedural work will take shape *after* the ideas are in place. Moreover, the shape of an individual student's procedural work will ultimately depend on how the student has formed connections in their network of concepts. It is therefore of vital importance that we first support students in building their notion of the core concepts before addressing any procedural issues.

It is also important to realize that, within the context of learning math as a language, it is not our role to teach students how to think, rather we need to recognize that each student may understand differently. Instead of a quick 'right' or 'wrong' when responding to students, we must recognize the diversity and potential of all contributions[10, 18]. While it may be tempting to intervene and provide quick aid by supplying strategies, procedures, and shortcuts [6, 13], we must abstain and instead reflect our students' ideas back to them for refinement[11].

5 Motivation

In general, the relevance of mathematical topics stems from their application to various problems. For instance, square-root expressions are necessary for the measurement of sides of right triangles, via the Pythagorean Theorem. The Pythagorean Theorem, and therefore its associated square-root expressions, also forms the basis for measuring the distance between points on the coordinate plane. Later on, square-roots play a role in the solution of quadratic equations, as well as in identifying various geometric features of other conic sections. In statistics, the measurement of standard deviation requires an understanding of the square root. In physics, the square root plays a frequent role in various calculations. Because expressing a number based on the value of its square is fundamental to many applications, we cannot settle for simply teaching students to "do" square-roots.

The proposed six stages of development towards mastery align with Alexander's Model of Domain Learning (MDL)[1]. The proposed Novice and Initiate levels of mastery correspond to Alexander's "acclimation" stage, where students typically have fragmented knowledge and are often limited in their individual, or preexisting, interest. Apprentice and Journeyman correspond to MDL's "competence" stage, where students show more interest and are less dependent on provided strategies. Finally, the Expert and Master levels correspond to Alexander's "proficiency" stage, where students are capable of both finding and solving problems that are "new".

When instructors teach mathematics through worked-out examples, they are effectively skipping the acclimation stage and instead moving directly into the MDL's "competence" stage. Sample work provided by instructors often "skips steps", as in the Apprentice level, emphasizing efficiency in the pursuit of providing more examples during their limited classroom hours, as in the Journeyman level. This approach leaves many students behind, as they have not progressed through the preceding levels. The MDL perspective suggests that this is the source of many students lack of interest in mathematics. Their learning needs are not being addressed and they remain in the acclimation stage with limited interest and knowledge.

Moreover, for those students who are "successful" with the traditional approach of teaching algorithms for problem-solving, instructors run the risk of stranding these students at the competence stage. These students sacrifice conceptual understanding for the sake of gaining procedural knowledge. The resulting loss of faith in their own conceptualization of mathematics yields an inflexibility in problem-solving, relying on fixed procedures without thought [8].

Developing procedures at the expense of concepts leads to rote memorization and may prevent the formation of a deeper understanding of the concept at hand [6]. By teaching mathematics as a language, we focus heavily on the acclimation stage of learning. Our efforts are primarily directed at the development of mathematical ideas, followed by a shift in emphasis towards literacy in the meta-langugage of mathematical symbols. The subsequent development of problem-solving skills (in the competence stage) is an exercise *for thes student* in constructing the relationships between the ideas and the symbols within the context of each problem. We, as instructors, should now shift to a more passive role, observing students' processes and reflecting back to them any apparent issues or misconceptions for further analysis and refinement.

All of this is not to say that students readily embrace the idea of learning mathematics as a language. There are many students who have developed the skill (and even the *expectation*) of learning math through mimicry. When an instructor begins to expect something deeper from their students, these individuals can feel as though the rug has been pulled out from underneath them. They have had a lot of practice in extrapolating a solution process from a series of examples. They felt comfortable in their ability to memorize a sequence of steps in order to solve a problem they had learned to recognize. These are survival skills, learned by students who endured high-stakes standardized exams from instructors who were under similarly high pressure to show student success.

These historical patterns and their associated expectations regarding teaching and learning forms what Brosseau described as the Didactic Contract [4]. This contract is implicit, rather than explicit, and an instructor who wishes to shift their learning environment away from procedural instruction towards one based on conceptual understanding must, in essence, renegotiate this contract. Establishing a learning environment based around the descriptive communication of mathematics takes time to develop and must be closely monitored and maintained. [15]. The effort required certainly pays off, as the Purdue Project [9] found that this kind of environment supported even the most conceptually immature student, leading him to feel as though his contributions were valued and he continued to contribute throughout the year. This environment of valuing of all students' contributions forms the foundation of a motivated and engaged classroom.

6 Mastery

As is the case with the development of students' learning during the competence stage, it is even more dependent on students' individual efforts through which proficiency can be attained. The evidence of students' mastery of a particular topic stems from their individual construction of relationships between internal ideas and their external representations. (This is why it's easy to tell when students cheat in mathematics - it is exceedingly rare that any two students will "show their work" in precisely the same way.) Individual differences in written work reflect the personal nature of this construction, and instructors should provide ample opportunity for students to share their unique processes with their peers. Again, our role in this is mainly supervisory, with the aim of recognizing and celebrating particularly distinct uses of the conceptual raw materials that we have provided. That is not to say that only those students who have constructed clever solutions should be sharing their work - we also have an opportunity to aid in students' development when they have less-than-efficient solutions. Any student work with redundant steps can provide a valuable teaching opportunity. Sharing this work and soliciting suggestions from other students engages the class in the necessary process of reflecting on one's own work. We not only help the individual student whose work we're revising, but we also help the class as a whole understand that problem-solving methods are flexible and that refining our process is both desirable and achievable.

In order to activate students' development of "proficiency" at the Expert or Master level, we must provide them with challenges that extend beyond the traditional problem-types found in textbooks and exam preparation materials. Neither finding nor constructing these challenging types of problems is easy. One immediate source of problems comes from future curriculum. Students who show evidence of being Journeymen in the area of simplifying numeric square roots are ready to be introduced to the challenges of simplifying algebraic square roots, or simplifying numeric roots with higher index (such as cube roots). Having additional "challenge" problems available sets more advanced students on a path of independent growth, while simultaneously allowing the instructor to focus their energy on other students who are still in the acclimation or competence stages.

Students working on the proficiency stage should be encouraged to work together on challenging problems. In attaining their current stage, they have displayed an ability to clearly communicate their thoughts, both as ideas and in the symbolic metalanguage of mathematics. Working with others ensures that they continue to develop these communication skills, and improves the flexibility of their conceptual relationships by being exposed to the same concepts as perceived by their peers.

Conclusion

What's vitally important in all of this is the instructor's belief in *all* students' ability to comprehend the fundamental principles of mathematics. This belief fuels their stubborn adherence to solving problems "the long way," insisting on the use of proper vocabulary at all times, and consistently requiring students to elaborate on their ways of thinking. As educators, our goal should be to produce students who have developed their own methods and procedures for solving various problems, rather than a whole classroom of students who all solve the same problems by following the same steps. Mathematics is way more resilient than that; and until we embrace the individual nature of our students' problem solving skills, I fear they will never embrace mathematics in return.

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